

Matrices and linear systems

Systems of equations and matrices that make sense: substitution, elimination, number of solutions, determinants and matrix products, with graphs and exam-style questions.

A system of two equations in two unknowns asks: **which pair (x, y) satisfies both at once?** Each equation is a line, and the solution is where they cross. If the lines are parallel there is no solution; if they are the same line there are infinitely many. A matrix is just a table holding the coefficients, and the **determinant** tells you whether the solution is unique.

Rule 1

Substitution and elimination

$$\begin{cases} x + y = 10 \\ x - y = 4 \end{cases} \Rightarrow 2x = 14$$

Substitution: solve for one letter and plug it into the other equation. **Elimination:** multiply equations so one letter cancels when you add. Always check the pair in **both** equations.

Rule 2

How many solutions?

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \rightarrow 1 \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \rightarrow 0$$

Different slopes: the lines cross, **one** solution. Same slope, different intercepts: parallel, **none**. Proportional equations: the same line, **infinitely many**.

Rule 3

Determinant and product

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$(AB)_{ij} = \text{row } i \text{ of } A \cdot \text{column } j \text{ of } B$$

Determinant $\neq 0$: the system has a unique solution. The product AB exists only if the number of columns of A equals the number of rows of B ; and in general $AB \neq BA$.

Worked examples, step by step

Worked example: Chickens and rabbits

A farm has chickens and rabbits, with 35 heads and 94 legs in total. How many rabbits are there?

1. $g = 35 - r$. Substituting: $2(35 - r) + 4r = 94$.
2. $70 + 2r = 94 \Rightarrow r = 12$.
3. $g = 23$. Check: $23 + 12 = 35$ heads and $46 + 48 = 94$ legs $\checkmark$.

Answer: 12 rabbits (and 23 chickens)

Worked example: The add-the-equations shortcut

If $2x + 3y = 17$ and $3x + 2y = 13$, find $x + y$, then $x - y$, and finally x and y .

1. Adding: $5(x + y) = 30 \Rightarrow x + y = 6$.
2. Second minus first: $x - y = 13 - 17 = -4$.
3. Adding $x + y = 6$ and $x - y = -4$: $2x = 2$, $x = 1$ and $y = 5$. Check: $2 + 15 = 17 \checkmark$.

Answer: $x + y = 6$, $x - y = -4$, $x = 1$, $y = 5$

Mistakes that cost points

Substituting without parentheses

× Wrong

$$x = y + 1 \text{ into } 2x + 3y = 12: \\ 2y + 1 + 3y = 12$$

✓ Right

$$\text{Substitute with parentheses} \\ \text{and distribute: } 2(y + 1) + \\ 3y = 2y + 2 + 3y.$$

Multiplying only one side

× Wrong

$$x + 2y = 5 \text{ times } 3: 3x + \\ 6y = 5$$

✓ Right

$$\text{Multiply every term, on both} \\ \text{sides: } 3x + 6y = 15.$$

Sign errors in the determinant

× Wrong

$$\det \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} = 8 - 3 = 5$$

✓ Right

$$ad - bc = 2 \cdot 4 - (-1) \cdot 3 = \\ 8 + 3 = 11.$$

Multiplying matrices entry by entry

× Wrong

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 21 & 32 \end{pmatrix}$$

✓ Right

$$\text{It is row times column:} \\ \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} \text{ (e.g. } 1 \cdot 5 + 2 \cdot 7 = \\ 19).$$