

Logarithms and exponentials

The essentials of logarithms and exponentials: get the idea, solve exam questions and avoid the mistakes that cost points.

A logarithm is the **reverse question of a power**. Instead of asking “what is 2^5 ?”, it asks “what exponent turns 2 into 32?”. That is why $\log_2 32 = 5$. Every logarithm problem starts by rewriting the log as a power.

Rule 1

Definition

$$\log_a b = c \iff a^c = b$$

For $a > 0$, $a \neq 1$ and $b > 0$. When in doubt, rewrite it as a power.

Rule 2

Product, quotient and power

$$\log(xy) = \log x + \log y$$

$$\log x^n = n \log x$$

Also $\log \frac{x}{y} = \log x - \log y$. Multiplication becomes addition; an exponent becomes a factor.

Rule 3

Change of base

$$\log_a b = \frac{\log b}{\log a}$$

Use it to compute any log on a calculator and to solve $a^x = b$: $x = \log_a b$.

Worked examples, step by step

Worked example: Logarithmic equation

Solve $\log_3(2x + 1) = 2$.

1. By the definition: $2x + 1 = 3^2 = 9$.
2. Solve for x : $2x = 8$, so $x = 4$.
3. Check the domain: $2(4) + 1 = 9 > 0$ ✓, and $\log_3 9 = 2$ ✓.

Answer: $x = 4$

Worked example: Bases that are powers of the same number

Find $\log_4 8$.

1. By the definition, $\log_4 8 = c$ means $4^c = 8$.
2. Rewrite in base 2: $(2^2)^c = 2^3$, that is, $2^{2c} = 2^3$.
3. Equal bases, equal exponents: $2c = 3$, so $c = \frac{3}{2}$.

Answer: $\frac{3}{2}$

Mistakes that cost points

Splitting the log of a sum

✗ Wrong

$$\log(a + b) = \log a + \log b$$

✓ Right

The rule is for a **product**: $\log(ab) = \log a + \log b$. The log of a sum does not simplify.

Raising the log instead of multiplying

✗ Wrong

$$\log x^n = (\log x)^n$$

✓ Right

The exponent **comes down as a factor**: $\log x^n = n \log x$.

Forgetting the domain

✗ Wrong

Solving the equation and keeping every root.

✓ Right

The argument of a log must be **positive**. Test every root in the original equation.

Mixing up a ratio of logs and the log of a ratio

✗ Wrong

$$\frac{\log a}{\log b} = \log \frac{a}{b}$$

✓ Right

$\log \frac{a}{b} = \log a - \log b$. And $\frac{\log a}{\log b}$ is a change of base: $\log_b a$.